

An Estimation of the Risk Premium as Function of the VIX

Introduction

The expected excess market return, or equity risk premium, is an unobservable element crucial in many financial applications ranging from corporate finance to asset pricing. Over the years, many researchers have studied the expected market return by tackling the estimation from different perspectives leading Welch, Goyal (2008) and Campbell, Thompson (2008) to re-examine the performance of such indicators from a common standpoint.

In the academic literature, the traditional predictors of the expected returns have been documented to be valuation ratios such as: dividend-price ratio, dividend yield, earnings-price ratio, or book-to-market ratio. More recently, Chabi-yo, Loudis (2019) suggests a different approach: the equity premium can be reliably quantified by conditional risk-neutral moments of excess market returns.

In this article, we take the Chabi-yo, Loudis (2019) lower bound on the conditional expected return as given, inviting the most curious readers to examine the original paper further, and solve an inherent practical problem of their work: the high computational intensity of the estimation. In fact, while the two authors use the Carr, Madan (2001) spanning formula in order to write risk-neutral moments in terms of option prices, we show how to simplify the process by approximating risk-neutral moments through functions of the VIX, a readily available index published by the Chicago Board Options Exchange.

Chabi-Yo, Loudis (2019) Lower Bound on the Conditional Excess Expected Market Return

Chabi-yo, Loudis (2019) lower bound is expressed as:

$$LBR_{t \rightarrow T} = \frac{\sum_{k=1}^3 \frac{(-1)^{k+1}}{R_f^k} M_{(k+1)}^*}{1 + \sum_{k=2}^3 \frac{(-1)^{k+1}}{R_f^k} M_{(k)}^*}$$

where $M_{(k)}^*$ stands for the conditional risk-neutral n-th excess moment:

$$M_{(n)}^* = E_t^*[(R_m - R_f)^n], \quad n > 0$$

and R_m, R_f for the market gross return and risk-free rate from t to T, where gross return is simply defined as S_T/S_t .

Lower Bound as Function of the VIX Index

According to its white paper, the VIX index is defined as the annualized risk-neutral expectation of integrated variance over the next 30 days:

$$VIX_t^2 = \frac{1}{T-t} E_t^* \left[\int_t^T \sigma_\tau^2 d\tau \right] = (\sigma_t^T)^2, \quad T = t + 30 \text{ days}$$

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In order to measure the VIX, the Cboe expresses the risk-neutral expectation in terms of a function of S_t , namely the logarithm.

By assuming a stock evolution process in the form of diffusion on the risk-neutral probability space (Ω, F, π^*) :

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t, \quad S_0 = s_0$$

where r_t and σ_t , respectively drift and volatility coefficients, are deterministic functions of time t ; and W_t is a standard Brownian motion on the same probability space. We apply Ito's lemma to obtain the stochastic differential equation (SDE) of the log-contract:

$$d \ln S_t = \left(r_t - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t, \quad \ln S_0 = \ln s_0$$

By taking the risk-neutral expectation of the integrated logarithm, we obtain:

$$\begin{aligned} E_t^* \left[\int_t^T d \ln S_\tau \right] &= E_t^* \left[\int_t^T \left(r_\tau - \frac{1}{2} \sigma_\tau^2 \right) d\tau + \int_t^T \sigma_\tau dW_\tau \right] \\ E_t^* \left[\ln \frac{S_T}{S_t} \right] &= r_t^T (T - t) - \frac{1}{2} E_t^* \left[\int_t^T \sigma_\tau^2 d\tau \right] \\ -2 E_t^* \left[\ln \frac{S_T}{F_t^T} \right] &= E_t^* \left[\int_t^T \sigma_\tau^2 d\tau \right] \\ -\frac{2}{T - t} E_t^* \left[\ln \frac{S_T}{F_t^T} \right] &= VIX_t^2 \end{aligned}$$

with $F_t^T = S_t R_f$. For our purposes, we are interested in using the VIX as an estimator of the risk-neutral expectation of log-return, thus we will be using the second line of the previous block of equations:

$$E_t^* [\ln R_m] = r_t^T (T - t) - \frac{1}{2} VIX_t^2 (T - t)$$

Under an assumption of log-normality of stock returns, also the risk-neutral 2nd return moment can be expressed as a function of VIX.

$$\begin{aligned} E_t^* [R_m^2] &= E_t^* \left[e^{2 \left[\int_t^T \left(r_\tau - \frac{1}{2} \sigma_\tau^2 \right) d\tau + \int_t^T \sigma_\tau dW_\tau \right]} \right] \\ &= e^{2r_t^T (T-t) - \int_t^T \sigma_\tau^2 d\tau} E_t^* \left[e^{\int_t^T 2\sigma_\tau dW_\tau} \right] \\ &= e^{2r_t^T (T-t) - (\sigma_t^T)^2 (T-t)} e^{0 + \frac{1}{2} [4(\sigma_t^T)^2 (T-t)]} \\ &= R_f^2 e^{(\sigma_t^T)^2 (T-t)}, \quad (\sigma_t^T)^2 = VIX_t^2 \end{aligned}$$

where on the third line we take the expectation of the exponential of the stochastic integral $\int_t^T 2\sigma_\tau dW_\tau$ which has mean 0 and variance $4(\sigma_t^T)^2 (T - t)$.

In order to approximate risk-neutral expectation of excess returns, we first decompose them into returns by breaking down the binomials. For example, for the 2nd excess moment we obtain:

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$$\begin{aligned} E_t^* \left[(R_m - R_f)^2 \right] &= E_t^* [R_m^2] + R_f^2 - 2E_t^* [R_m] R_f \\ &= E_t^* [R_m^2] - R_f^2 \end{aligned}$$

where the risk-neutral expectation of gross return is the risk-free factor.

Then, we proceed by following the intuition of Bakshi, Kadapia, Madan (2003): the expectation of the gross return, R_m , can be decomposed into expectation of log-returns through a Taylor expansion on $f(\ln R_m) = e^{\ln R_m} = R_m$ around $\ln R_m = 0$. In addition, such intuition can be applied to all powers of gross returns.

In general, let $x = \ln R_m$ be our independent variable, and $f(x) = e^{nx}$ be our function of interest; by taking the Taylor expansion around $x_0 = 0$, and up to the 3rd order, we obtain:

$$e^{nx} = 1 + nx + \frac{1}{2}n^2x^2 + \frac{1}{6}n^3x^3 + o(x^4)$$

by writing the equation in terms of gross returns R_m and taking its risk-neutral expectation:

$$e^{nx} = 1 + nx + \frac{1}{2}n^2x^2 + \frac{1}{6}n^3x^3 + o(x^4)$$

Then, we observe that some of the terms within this equation are already known: $E_t^* [R_m]$ is equal to the risk-free factor, $E_t^* [\ln R_m]$ and $E_t^* [R_m^2]$ are both functions of VIX. Thus, by solving a system of 2 equations above (for $n=1,2$) with 2 unknowns allows us to find, first, the missing log-return moments, and then the interested return moments (for $n=3,4$).

$$\begin{cases} E_t^* [(\ln R_m)^2] = \frac{1}{2} [8R_f - 6E_t^* [\ln R_m] - E_t^* [R_m^2] - 7] \\ E_t^* [(\ln R_m)^3] = \frac{3}{2} [-4R_f + 2E_t^* [\ln R_m] + E_t^* [R_m^2] + 3] \end{cases}$$

The last step remains to reconstruct the binomials of excess return from return moments.

Application to the US Market

In the previous section, we demonstrated that the Chabi-yo, Loudis (2019) lower bound can be replicated efficiently by a function of the VIX index. This allows practitioners to quickly obtain an estimate of the conditional expected market return which can be computed at daily or even higher frequency.

In this section, we apply such methodology on the US market, proxied by the S&P 500 index. While the 30d risk-free rates are obtained from OptionMetrics, the SPX and VIX indices are obtained from Bloomberg. The sample period runs from 1st Jan 1996 to 31st Dec 2018.

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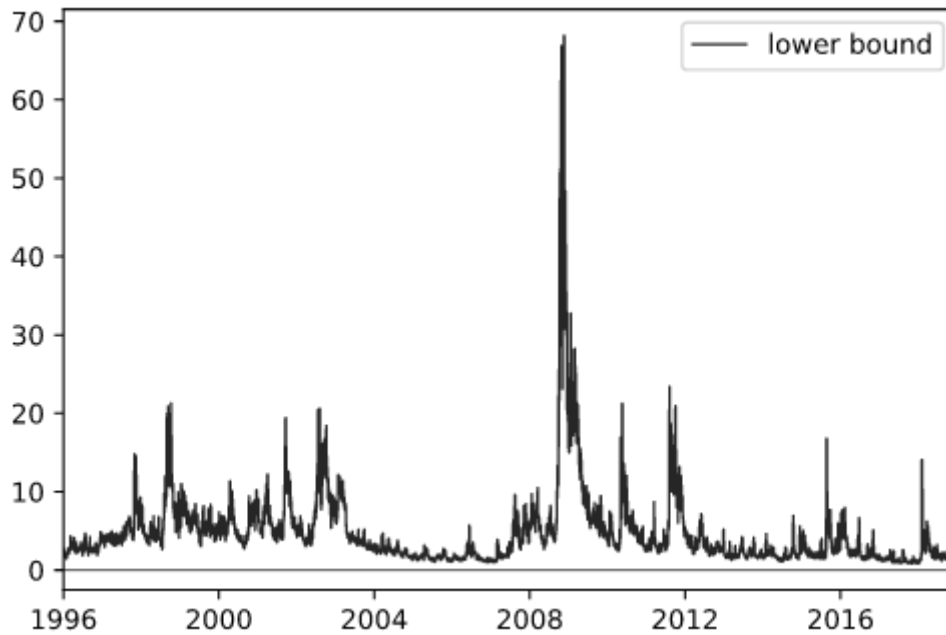


Figure 1: Lower bound of the conditional expected market return, in annualized and percentage terms.

In figure 1 we plot the time-series of the replicated 30d lower bound (annualized and in percentage terms), which can be seen to be highly volatile and tends to spike during periods of economic stress.

In order to show that such bound is a good predictor of the risk premium, we run an OLS forecasting regressions of the following form:

$$R_{m,t \rightarrow T} - R_{f,t \rightarrow T} = a + bLB_{t \rightarrow T} + \epsilon$$

In addition, we compute the out-of-sample R^2 of Welch, Goyal (2008) and Campbell, Thompson (2008) by following the predicting expanding window regression of Ferreira, Santa-Clara (2011). In their paper, they explain how they simulate what a forecaster could have done in real time.

Specifically, we set the out-of-sample date as $s_0 = 31\text{st Dec } 2016$, i.e. 2 years before the end of sample. Next, we split the sample into 2 sub-groups: an expanding in-sample of $t = 1, 2, \dots, s$ and contracting sample of $t = s + 1, s + 2, \dots, S$ with the varying end-point $s \in [s_0, S]$. Note the difference between $T = t + 30d$ and S being the last date in our sample, i.e. 31st Dec 2018.

Using the in-sample observations, we estimate forecasting regression to obtain a forecast for $t = s + 1$ as conditional expectation:

$$E_s[R_{m,s+1} - R_{f,s+1}] = \widehat{a}_s + \widehat{b}_s x_s$$

where we denote with the subscript s the estimated parameters of the forecasting regression with the sample up to s .

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Such estimators are compared against the historical excess mean model estimated through the difference between the geometric average return of the market and of the risk-free rate from CRSP beginning in 1926.

Finally, we measure the forecasting performance of our variable against the historical sample mean using the out-of-sample R^2 computed according to:

$$R_{OOS}^2 = 1 - \frac{MSE_A}{MSE_N}$$

where MSE_A is the mean square error from the predicting regression and MSE_N from the historical mean model:

$$MSE = \frac{1}{T} \sum_{s=1}^T (Y_s - \hat{Y}_s)^2$$

It must be stated that while the standard R^2 can take values from 0 to 1, the out-of-sample one can take negative values. This happens when the $MSE_N < MSE_A$, i.e. the historical mean model predicts better than the interested regression.

	Days	a	t(a)	b	t(b)	R2 (%)	OOS R2 (%)
Regression	30	0.04	0.92	0.38	0.46	0.13	0.88

Table 1: Regression results of the lower bound on the historical excess market return, annualized for the period starting 1st Jan 1996 to 31st Dec 2018, with an out-of-sample starting date of 31st Dec 2016. The historical mean model for the out-of-sample R^2 uses information from Jan 1926.

Table 1 reports the result of such approach. From it, we can notice that the intercept and slope coefficient cannot be rejected to be statistically different from 0, 1. It entails that the bound is fairly tight.

The out-of-sample R^2 statistic of 0.88% shows promising result, especially if compared with the findings of Welch, Goyal (2008) which test many conventional predictor variables as such: dividend-price ratio, dividend yield, earnings-price ratio, book-to-market ratio and obtain negative out-of-sample R^2 of -2.06%, -1.93%, -1.78% and -1.72% respectively.

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